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BIFURCATIONS IN ELASTIC–PLASTIC MATERIALS

M. K. NEILSEN
Engineering Mechanics and Material Modeling—1561, Sandia National Laboratories, Albuquerque, NM 87185, U.S.A.

and

H. L. SCHREYER
Department of Mechanical Engineering, University of New Mexico, Albuquerque, NM 87131, U.S.A.

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Abstract—When strain-softening, elastic–plastic materials are loaded into the plastic regime they often exhibit deformations that are localized into small regions at some point in the loading process. This intense localized deformation limits the formability of materials and will often quickly lead to failure with continued loading. Localized deformation is often associated with satisfaction of the classical discontinuous bifurcation criterion. Here we propose that the loss of strong ellipticity criterion should be used in place of the classical discontinuous bifurcation criterion as a necessary condition for localization. The application of the strong ellipticity criterion implies that a bifurcation mode associated with loss of positive definiteness of the symmetric part of the acoustic tensor must be identified rather than a mode associated with the first zero eigenvalue of the acoustic tensor itself. The eigensystem for the symmetric part of the tangent modulus tensor is obtained for several different plasticity models. This eigensystem provides information about deformation modes associated with both diffuse and discontinuous bifurcations. Material properties, boundary conditions and body geometry are all shown to affect the diffuse and localized deformation modes that are generated. Numerous experimental observations of necking and localization in metal specimens subject to various boundary conditions are explained with the proposed approach.

INTRODUCTION

Necessary but not sufficient conditions for diffuse and discontinuous bifurcations, loss of uniqueness, and localized deformation of elastic–plastic materials have been previously developed. A necessary condition for loss of material stability, loss of uniqueness and any bifurcation in the solution is the loss of positive definiteness of the rate of second order work (Drucker, 1950; Hill, 1958). This general bifurcation criterion can also be expressed as loss of positive definiteness of the symmetric part of the tangent modulus tensor. The eigentensor, or mode, associated with a general bifurcation may or may not have a kinematically compatible form. A mode which is not kinematically compatible, can only exist in a zone described as a point or surface, i.e. a domain of measure zero. Such a mode initiates smooth changes in the deformation field such as necking and is usually referred to as a diffuse mode.

Valanis (1989) states that loss of material stability should be associated with the limit point where the tangent modulus tensor obtains a zero eigenvalue. This may be appropriate if one considers only statically determinant specimens with force prescribed systems. For materials with symmetric tangent modulus tensors, the Valanis (1989) and Drucker (1950) interpretations both identify the limit point as the point at which necessary conditions for loss of material stability are first satisfied. For materials with unsymmetric tangent modulus tensors, loss of positive definiteness of the symmetric part of the tangent modulus tensor and satisfaction of the necessary condition for a general bifurcation can occur prior to the limit point.

Hill (1962), Mandel (1966), Rudnicki and Rice (1975) and Rice (1976) have stated that loss of material stability and localization will not occur until the acoustic tensor obtains a zero eigenvalue. The acoustic tensor is dependent on both an orientation vector and on the material. Localization is associated with a strain rate jump within a planar band that does not lead to any kinematic incompatibilities with the surrounding material. Since the mode can be interpreted as a jump in strain rate within a band over the strain rate in the surrounding material, the mode is called a discontinuous bifurcation. Localization may initiate on a domain indefinite in extent. Evolution of the domain can only be controlled if additional constraints such as those provided by a non-local constitutive theory are present (Schreyer, 1990). A spectral analysis of the acoustic tensor is rather difficult, although recent work in this area by Ottosen and Runesson (1991), Bigoni and Hueckel (1991) and Runesson *et al.* (1991) appears encouraging.

Ottosen and Runesson (1991) state that loss of strong ellipticity of the governing differential equations occurs whenever positive definiteness of the symmetric part of the acoustic tensor is lost. When the tangent modulus tensor and thus the acoustic tensor are symmetric, loss of strong ellipticity and satisfaction of the classical necessary condition for a discontinuous bifurcation will first occur at the same point. However, for non-symmetric acoustic tensors, loss of strong ellipticity will precede satisfaction of the necessary condition for a classical discontinuous bifurcation. In addition to the loss of positive definiteness of the acoustic tensor, loss of strong ellipticity can also be interpreted as satisfaction of the general bifurcation condition with an associated strain rate which is of a form suitable for providing a kinematically compatible velocity field (Bigoni and Hueckel, 1991). Here we postulate that localization should be associated with the loss of strong ellipticity.

An eigenanalysis of the symmetric part of the tangent modulus tensor provides a wealth of information about deformation modes associated with both diffuse and discontinuous bifurcations. The necessary condition for a general bifurcation is first satisfied when the fundamental eigenvalue of the symmetric part of the tangent modulus tensor obtains a value of zero. The deformation mode associated with this bifurcation is characterized by the fundamental eigentensor. Discontinuous bifurcations are investigated by choosing a mode which satisfies the general bifurcation criterion and is restricted to be of a form normally associated with a discontinuous bifurcation. The mode is expressed as a linear combination of the eigentensors associated with the symmetric part of the tangent modulus tensor. With this approach there is no need to explicitly determine or analyse the acoustic tensor. By interpreting the problem of material stability as an eigensystem problem in the presence of a constraint, we automatically establish the structure for incorporating additional constraints, such as plane strain, which may be present because of external loading and boundary conditions. The presence of additional constraints automatically infer that the usual procedure of finding the first zero eigenvalue of the acoustic tensor may not be an indicator of localization.

The approach used is to determine the spectral decomposition of the symmetric part of the tangent modulus tensor. For conventional plasticity models, this decomposition is straightforward and explicit linear combinations of eigentensors can be obtained to show satisfaction of constraints which reduce to linear algebraic equations of the eigentensors. Some of the constraints can only be satisfied if the fundamental eigenvalue is negative which only occurs when plasticity models with associated flow rules exhibit strain-softening. The degree of softening, if any, required to meet the constraint condition depends on the particular loading path being considered.

With the insight provided by this approach, we show that features exhibited by a number of classical experiments can be easily explained using simple constitutive models. The necking and localization of a material subject to various constraints can provide valuable confirmation of the suitability of a constitutive model. For simplicity, we confine our attention to rate and temperature independent material behavior and infinitesimal deformations.

GENERAL BIFURCATION

Drucker (1950) postulated that the stability of a material could be evaluated by considering the work done by an external agency. A material is stable (will remain in equilibrium) if (a) positive work is done by the external agency during the application of the set of stresses and (b) the net work done by it over a cycle of application and removal is zero or positive. If plastic deformation is generated during the cycle then the net work must be non-zero. These statements indicate that a necessary condition for loss of material stability is

$$\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\sigma}} = 0, \tag{1}$$

where $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$ are the stress and strain rates at some point or region in the body. Equation (1) was shown by Hill (1958) to be a necessary condition for any type of bifurcation and loss of uniqueness. This general bifurcation criterion, eqn (1), can also be written as

$$\dot{\boldsymbol{\varepsilon}} : \mathbf{D}^{s} : \dot{\boldsymbol{\varepsilon}} = 0, \tag{2}$$

where $\mathbf{D}^{s}$ is the symmetric part of the tangent modulus tensor, $D^{s}_{ijkl} = \frac{1}{2}(D_{ijkl}+D_{klij})$. All of the fourth order tensors used in this paper possess the minor symmetries $D_{ijkl} = D_{jikl}$ and $D_{ijkl} = D_{ijlk}$. Equation (2) indicates that general bifurcations may occur whenever $\mathbf{D}^{s}$ is not positive definite.

LIMIT POINT BIFURCATION

General bifurcations are usually associated with non-zero stress rates both inside and ouside the bifurcation zone. The subset of general bifurcations associated with a null stress rate occur only at the limit point when

$$\mathbf{D} : \dot{\boldsymbol{\varepsilon}} = 0 \quad \text{or} \quad \det(\mathbf{D}) = 0, \tag{3}$$

or, in other words, when the tangent modulus tensor $\mathbf{D}$ has a zero eigenvalue. Valanis (1989) recently suggested that eqn (3) is a necessary and sufficient condition for loss of material stability. His interpretation assumes that the general bifurcation associated with the limit point is always activated. If constraints are present, the general bifurcation associated with the limit point may not be activated, and it becomes necessary to evaluate other potential general and discontinuous bifurcations. Here we adopt the classical necessary condition for a general bifurcation given by eqn (2).

CLASSICAL DISCONTINUOUS BIFURCATION

A criterion for discontinuous bifurcations in elastic–plastic materials with associated flow rules follows from Hadamard's (1903) studies of elastic stability and was developed by Hill (1962). Later, Mandel (1966), Rice (1976), Rudnicki and Rice (1975), Rice and Rudnicki (1980), Raniecki and Bruhns (1981) and Ottosen and Runesson (1991) used Hill's criterion to investigate discontinuous bifurcations in elastic–plastic materials with both associated and non-associated flow rules.

Consider a homogeneous solid subjected to monotonic, proportional loading. We wish to determine at what point in the loading process a discontinuous bifurcation can occur such that subsequent strain rates become discontinuous across parallel planes of orientation $\mathbf{n}$ that separate a zone of localized deformation from the rest of the body. Maxwell's compatibility conditions require that the strain rate in the localized zone, $\dot{\boldsymbol{\varepsilon}}^{i}$, be of the form

$$\dot{\boldsymbol{\varepsilon}}^{i} = \dot{\boldsymbol{\varepsilon}}^{o} + \dot{\boldsymbol{\varepsilon}}^{k} \quad \text{with} \quad \dot{\boldsymbol{\varepsilon}}^{k} = \tfrac{1}{2}(\dot{\mathbf{m}} \otimes \mathbf{n} + \mathbf{n} \otimes \dot{\mathbf{m}}), \tag{4}$$

where $\dot{\boldsymbol{\varepsilon}}^{o}$ is the strain rate outside the localized zone, $\dot{\boldsymbol{\varepsilon}}^{k}$ is a kinematically admissible discontinuous mode, and $\dot{\mathbf{m}}$ can be interpreted as a vector that represents the orientation of the relative velocity of regions on opposite sides of the localized deformation zone due to the introduction of the localized zone. The discontinuous bifurcation mode $\dot{\boldsymbol{\varepsilon}}^{k}$ is characterized by the following eigenvalues: $\lambda_1 \leqslant 0$, $\lambda_2 = 0$ and $\lambda_3 \geqslant 0$ with at least one eigenvalue being non-zero.

Assume that the entire body is being plastically deformed, the stress and strain components are uniform throughout, and the body is at the onset of localization. With the assumption of rate-independent behavior, the stress rates inside and outside the localized zone are given by

$$\dot{\boldsymbol{\sigma}}^{i} = \mathbf{D}^{i} : \dot{\boldsymbol{\varepsilon}}^{i} \quad \text{and} \quad \dot{\boldsymbol{\sigma}}^{o} = \mathbf{D}^{o} : \dot{\boldsymbol{\varepsilon}}^{o}, \tag{5}$$

where $\mathbf{D}^{i}$ and $\mathbf{D}^{o}$ are the tangent modulus tensors for material inside and outside the localized deformation zone, respectively. For continuing equilibrium, the traction rates must be continuous across the boundaries of the localized deformation zone:

$$\dot{\mathbf{t}}^{i} = \dot{\mathbf{t}}^{o} \quad \text{or} \quad \mathbf{n}\cdot(\dot{\boldsymbol{\sigma}}^{i} - \dot{\boldsymbol{\sigma}}^{o}) = \mathbf{0}. \tag{6}$$

By combining these equations, Rice (1976) shows that the requirement for continuing equilibrium is given by

$$\mathbf{n}\cdot(\mathbf{D}^{i} - \mathbf{D}^{o}) : \dot{\boldsymbol{\varepsilon}}^{o} + \mathbf{Q}\cdot\dot{\mathbf{m}} = \mathbf{0}, \tag{7}$$

where

$$\mathbf{Q} = \mathbf{n}\cdot\mathbf{D}^{i}\cdot\mathbf{n} \tag{8}$$

is the acoustic tensor.

Suppose the body is loaded such that the strain rate $\dot{\boldsymbol{\varepsilon}}^{o}$ is constrained to evolve continuously. Then it is reasonable to assume that the tangent modulus tensor for material outside the localized zone, $\mathbf{D}^{o}$, is identical to the tangent modulus tensor for material inside the localized zone, $\mathbf{D}^{i}$, at the initiation of the bifurcation. The classical necessary condition for a discontinuous bifurcation is then obtained from eqn (7):

$$\mathbf{Q}\cdot\dot{\mathbf{m}} = \mathbf{0} \quad \text{or} \quad \det(\mathbf{Q}) = 0. \tag{9}$$

In other words, the classical criterion for a discontinuous bifurcation is that the acoustic tensor, $\mathbf{Q}$, has a zero eigenvalue, a necessary condition for loss of ellipticity (Rice, 1976).

LOSS OF STRONG ELLIPTICITY

The classical discontinuous bifurcation criterion is based on two important assumptions. The first assumption is that the discontinuity in the strain rate field is constrained to have a special form so that material in the localized zone will remain kinematically compatible with the surrounding material. The second assumption is that the strain rates evolve continuously such that the tangent modulus tensor for material inside the localized zone is identical to the tangent modulus tensor for material outside the localized zone during the initiation of the localized zone. The general bifurcation criterion requires neither of these assumptions. Specifically, a general bifurcation will not necessarily be associated with a mode which has the special form of $\dot{\boldsymbol{\varepsilon}}^{k}$ in eqn (4) and the active tangent modulus tensors for material inside and outside the bifurcation zone will not necessarily be identical.

The general bifurcation criterion [eqn (2)] is a necessary condition for any type of bifurcation. A necessary condition for a general bifurcation with a kinematically compatible mode, $\dot{\boldsymbol{\varepsilon}}^{k}$, is the loss of strong ellipticity criterion (Bigoni and Hueckel, 1991)

$$\dot{\boldsymbol{\varepsilon}}^{k} : \mathbf{D}^{s} : \dot{\boldsymbol{\varepsilon}}^{k} = 0 \quad \Rightarrow \quad \dot{\mathbf{m}}\cdot\mathbf{Q}^{s}\cdot\dot{\mathbf{m}} = 0. \tag{10}$$

We adopt loss of strong ellipticity as a necessary condition for localization because this criterion identifies the first possible bifurcation with a kinematically compatible mode.

The requirement for continuing equilibrium [eqn (7)] may be satisfied when loss of strong ellipticity occurs if the continuity constraint (in time) on $\dot{\boldsymbol{\varepsilon}}^{o}$ is relaxed. For example, let $\mathbf{Q}$ be decomposed into its symmetric, $\mathbf{Q}^{s}$ and antisymmeric, $\mathbf{Q}^{u}$, parts:

$$\mathbf{Q} = \mathbf{Q}^{s} + \mathbf{Q}^{u}. \tag{11}$$

Loss of strong ellipticity [eqn (10)] will first occur when

$$\mathbf{Q}^{s}\cdot\dot{\mathbf{m}} = 0 \quad \text{or} \quad \det(\mathbf{Q}^{s}) = 0, \tag{12}$$

since $\mathbf{Q}^s$ is positive, semi-definite at this point. This criterion [eqn (12)] will be satisfied prior to or at the same time as the classical discontinuous bifurcation criterion of eqn (9). If no external constraint is placed on $\dot{\boldsymbol{\varepsilon}}^{o}$ other than compatibility, then $\dot{\boldsymbol{\varepsilon}}^{o}$ is free to adjust such that the continuing equilibrium equation

$$\mathbf{n}\cdot(\mathbf{D}^{i}-\mathbf{D}^{o}):\dot{\boldsymbol{\varepsilon}}^{o}+\mathbf{Q}^{u}\cdot\dot{\mathbf{m}} = \mathbf{0} \tag{13}$$

is satisfied when the loss of strong ellipticity criterion is satisfied. This means that a discontinuous bifurcation may occur when the loss of strong ellipticity criterion is satisfied and $\dot{\boldsymbol{\varepsilon}}^{o}$ is not constrained.

SUMMARY OF BIFURCATION CRITERIA

The criteria for diffuse and discontinuous bifurcations are summarized in Table 1. The general bifurcation criterion is first satisfied when the determinant of the symmetric part of the tangent modulus tensor is equal to zero. For materials with associated flow, the tangent modulus tensor is symmetric and the general and limit point bifurcation criteria both identify the limit point as the first point at which any type of bifurcation may occur. However, for materials with non-associated flow the general bifurcation criterion indicates that bifurcations may occur in the hardening regime.

The loss of strong ellipticity criterion is first satisfied when the determinant of the symmetric part of the acoustic tensor is equal to zero. For materials with associated flow rules, the acoustic tensor is symmetric and the loss of strong ellipticity and classical discontinuous bifurcation criteria identify the same first discontinuous bifurcation point. However, for materials with non-associated flow, the loss of strong ellipticity criterion will predict that localization may occur prior to the point identified by the classical discontinuous bifurcation criterion.

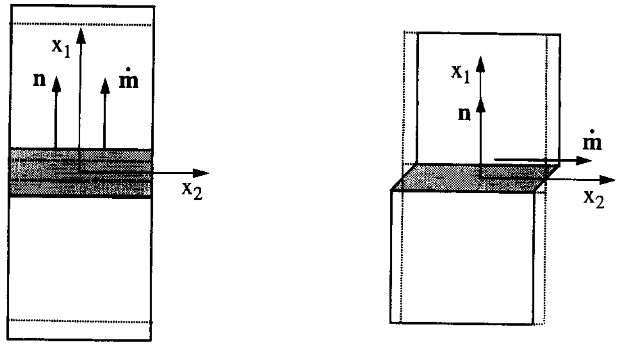
Discontinuous bifurcations are a subset of general bifurcations and classical discontinuous bifurcations are a subset of those discontinuous bifurcations which satisfy the loss of strong ellipticity criterion. Constraints may inhibit the activation of certain possible bifurcation modes for which the necessary but not sufficient conditions given in the previous section have been satisfied.

CHARACTERIZATION OF BIFURCATION MODES

Bifurcation modes, $\dot{\boldsymbol{\varepsilon}}$, represent perturbations to a homogeneous strain rate field that may be activated whenever the necessary conditions presented in the previous section are satisfied. Any bifurcation mode, $\dot{\boldsymbol{\varepsilon}}$, can be characterized by its three eigenvalues, $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$. Modes associated with discontinuous bifurcations are restricted to be of the kinematically compatible form, $\dot{\boldsymbol{\varepsilon}}^{k}$, given in eqn (4). Suppose we choose a local coordinate system with coordinate x_1 parallel and coordinates x_2 and x_3 perpendicular to $\mathbf{n}$ such that the components of $\mathbf{n}$ and $\dot{\mathbf{m}}$ are

Table 1. Summary of bifurcation criteria

Criterion	Equation	Mode
General	$\dot{\boldsymbol{\varepsilon}}:\mathbf{D}^{s}:\dot{\boldsymbol{\varepsilon}} = 0$	Diffuse or localized
Limit point	$\mathbf{D}:\dot{\boldsymbol{\varepsilon}} = 0$	Diffuse or localized
Loss of strong ellipticity	$\dot{\mathbf{m}}\cdot\mathbf{Q}^{s}\cdot\dot{\mathbf{m}} = 0$	Localized
Classical discontinuous	$\mathbf{Q}\cdot\dot{\mathbf{m}} = 0$	Localized



(a) (b)

Fig. 1. Discontinuous bifurcation modes: (a) opening, (b) shearing.

$$\mathbf{n} \Rightarrow \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \dot{\mathbf{m}} \Rightarrow \begin{Bmatrix} \alpha \\ 2\beta \\ 0 \end{Bmatrix}. \tag{14}$$

The components of the corresponding discontinuous bifurcation mode, $\dot{\varepsilon}^{\mathrm{k}}$, are

$$\dot{\varepsilon}^{\mathrm{k}} \Rightarrow \begin{bmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

When β is equal to zero, the relative velocity of the bodies is oriented in a direction normal to the zone, and the strain rate jump in the zone represents an opening mode (Fig. 1). When α is equal to zero, the strain rate jump represents a shearing mode. Also, note that $\dot{\varepsilon}^{\mathrm{k}}$ has eigenvalues of $\alpha/2 \pm \sqrt{\alpha^2/4+\beta^2}$ and zero. Thus, a discontinuous bifurcation mode has a fundamental eigenvalue, λ_1, that is less than or equal to zero, an intermediate eigenvalue, λ_2, that is equal to zero, and a third eigenvalue, λ_3, that is greater than or equal to zero.

Modes associated with general bifurcations can be any symmetric second order tensor as long as the necessary condition for a general bifurcation [eqn (2)] is satisfied. For example, a general bifurcation mode could have components obtained as a slight generalization of eqn (15):

$$\dot{\varepsilon} \Rightarrow \begin{bmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}. \tag{16}$$

An analysis in the x_1–x_2 plane can be performed as before for a discontinuous bifurcation with the origin of x_3 at the surface of a possible discontinuity; however, now an incompatibility in the velocity field exists for points $x_3 \neq 0$ which are not in the x_1–x_2 plane (Fig. 2). There are two ways to interpret this situation. In brittle materials, some experimental specimens exhibit microcracking in a specific orientation which could be considered a manifestation of the incompatible velocity field. In ductile materials, the potential development of an incompatible mode will initiate smooth changes in the deformation field such as necking.

The construction given above displays a compatible mode in the x_1–x_2 plane with the possibility of an incompatible component in the x_3 direction. Of course, a compatible mode could possibly exist in other planes. The actual orientation of the necked region might be based on the geometry of the specimen. For example, if the dimensions of a specimen in

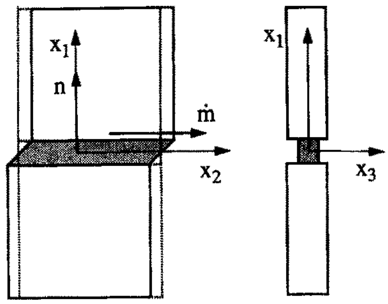


Fig. 2. A diffuse bifurcation mode.

the x_1 and x_2 directions are much larger than the dimension in the x_3 direction, then there might be a preference for most materials to neck with the direction of the potential incompatibility oriented towards the minimum dimension of the specimen.

In general, for a localized zone to form and remain compatible in some plane with the surrounding material, there must be some orientation in the plane such that the normal component of the localized mode is equal to zero. In terms of the eigenvalues associated with a bifurcation mode, compatibility can only exist in principal planes for which $\lambda_i\lambda_j \leqslant 0$. Recall, that a discontinuous bifurcation mode has a zero intermediate principal value and is thus compatible in all three principal planes. Diffuse bifurcation modes with one positive and two negative eigenvalues are compatible only in the principal planes with eigenvalues of opposite sign. Finally, diffuse bifurcation modes that are positive or negative definite are compatible only at a single point and are not compatible in any principal plane.

BIFURCATION CRITERIA AND THE EIGENSYSTEM FOR $\mathbf{D}^s$

In this section, we investigate the relationship between the bifurcation criteria presented in the previous section and the eigensystem associated with the symmeric part of the tangent modulus tensor, $\mathbf{D}^s$. Consider the eigenvalue problem

$$\mathbf{D}^s : \mathbf{x}_i = \omega_i \mathbf{x}_i, \tag{17}$$

in which $\mathbf{x}_i$ denotes an eigentensor for $\mathbf{D}^s$ and ω_i the corresponding eigenvalue. Due to its minor symmetries, $\mathbf{D}^s$ has six symmetric and three skew-symmetric eigentensors. Since the bifurcation modes, $\dot{\boldsymbol{\varepsilon}}$, are symmetric second-order tensors, we confine our attention to only the symmetric eigentensors associated with $\mathbf{D}^s$ and refer to them simply as the eigentensors associated with $\mathbf{D}^s$ throughout the remainder of this paper.

For convenience, we normalize the eigentensors so that

$$\mathbf{x}_i : \mathbf{x}_j = \delta_{ij} \tag{18}$$

and order the eigenvalues such that $\omega_1 \leqslant \omega_2 \leqslant \ldots \omega_6$. A bifurcation mode, or any symmetric second-order tensor, can be written as a linear combination of the eigentensors associated with $\mathbf{D}^s$ as follows:

$$\dot{\boldsymbol{\varepsilon}} = \sum_{i=1}^{6} \alpha_i \mathbf{x}_i \tag{19}$$

and the necessary condition for a general bifurcation from eqn (2) can be written as

$$\dot{\boldsymbol{\varepsilon}} : \mathbf{D}^s : \dot{\boldsymbol{\varepsilon}} = \sum_{i=1}^{6} \alpha_i^2 \omega_i = 0. \tag{20}$$

The necessary condition for a general bifurcation is first satisfied when $\omega_1 = 0$, and the corresponding bifurcation mode is characterized by the fundamental eigentensor. If constraints from the geometry or boundary conditions are present, the bifurcation mode given by the fundamental eigentensor may not be activated. If the material strain softens with continued loading, the fundamental eigenvalue will become negative and the necessary condition [eqn (20)] for numerous other diffuse and discontinuous bifurcation modes will be satisfied. A discontinuous bifurcation associated with loss of strong ellipticity may be activated when the general bifurcation criterion is satisfied and the corresponding $\dot{\varepsilon}$ has a special form. Specifically, the intermediate eigenvalue for this second order tensor, $\dot{\varepsilon}$, must equal zero. Finally, localization will only occur when the necessary condition for a discontinuous bifurcation mode that is not constrained by the boundary conditions is satisfied.

EIGENANALYSIS OF THE ELASTIC TANGENT MODULUS TENSOR

Eigenanalyses of the tangent modulus tensors provide a wealth of information about both diffuse and discontinuous bifurcation modes and aid in the identification of constrained bifurcation modes. Here, it is shown that the eigensystem for an elastic tangent modulus tensor can be easily obtained. In subsequent sections, eigensystems for plastic tangent modulus tensors are obtained.

For an elastic increment in an isotropic material, the tangent modulus tensor is the elasticity tensor, $\mathbf{E}$, given by

$$\mathbf{E} = 3K\mathbf{P}^{sp} + 2G\mathbf{P}^{d}, \tag{21}$$

where K is the bulk modulus and G is the shear modulus for the elastic material. The bulk and shear moduli are related to Young's modulus, E, and Poisson's ratio, ν, as follows:

$$K = \frac{E}{3(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}. \tag{22}$$

The fourth order spherical projection operator, $\mathbf{P}^{sp}$, and the deviatoric projection operator, $\mathbf{P}^{d}$, are given by

$$\mathbf{P}^{sp} = \tfrac{1}{3}\mathbf{i} \otimes \mathbf{i}, \quad \mathbf{P}^{d} = \mathbf{I} - \mathbf{P}^{sp}. \tag{23}$$

Here $\mathbf{I}$ is the symmetric fourth order identity tensor and $\mathbf{i}$ is the second order identity. The spherical projection operator $\mathbf{P}^{sp}$, has only one nonzero eigenvalue of one with a corresponding eigentensor equal to the second order identity. All of the other symmetric eigentensors for $\mathbf{P}^{sp}$ are in a deviatoric space, a space of symmetric second order tensors orthogonal to $\mathbf{i}$. The deviatoric projection operator, $\mathbf{P}^{d}$, has an eigenvalue of one with a multiplicity of five. The corresponding five eigentensors are orthogonal to $\mathbf{i}$ and span the deviatoric space. The second order identity is also an eigentensor for $\mathbf{P}^{d}$ with a corresponding eigenvalue of zero. With this information and the expression for $\mathbf{E}$, one observes that $\mathbf{E}$ has an eigenvalue of $3K$ with a multiplicity of one and an eigenvalue of $2G$ with a multiplicity of five. The corresponding eigentensors are the second order identity and any set of five tensors which are orthogonal to the identity and span the deviatoric space, respectively. Specifically, the components in a Cartesian coordinate system of the normalized eigentensors, $\mathbf{e}_i$, for $\mathbf{E}$ can be chosen to be the following:

$$\frac{1}{\sqrt{6}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

with corresponding eigenvalues of $\lambda_i = 2G$, $i = 1,5$ and $\lambda_6 = 3K$. The set of eigentensors given above is not unique; however, the set must always span the space of symmetric second-order tensors. Also, note that if Poisson's ratio equals zero then $2G = 3K = E$ and all of the eigenvalues are equal to E and $\mathbf{E} = E\mathbf{I}$. Since the elastic tangent modulus tensor, $\mathbf{E}$, is symmetric and positive definite, bifurcations cannot occur during elastic loading or unloading.

EIGENANALYSIS OF PLASTIC TANGENT MODULUS TENSOR

A plasticity model is characterized by a yield function, Ψ, which defines a surface in stress space separating elastic and plastic regimes and a flow rule

$$\dot{\varepsilon}^{\mathrm{p}} = \dot{\rho}\mathbf{g}, \tag{25}$$

where $\mathbf{g}$ is a second order tensor which defines the orientation of the plastic strain increment and ρ is a monotonically increasing parameter. The tangent modulus tensor when plasticity is occurring is given by

$$\mathbf{D} = \mathbf{E} - \frac{1}{A}\mathbf{E}:\mathbf{g}\otimes\mathbf{f}:\mathbf{E}, \tag{26}$$

where $\mathbf{f}$ is the normal to the yield surface defined by the yield function Ψ:

$$\mathbf{f} = \frac{\partial\Psi}{\partial\boldsymbol{\sigma}}. \tag{27}$$

The scalar A is given by

$$A = H + \mathbf{g}:\mathbf{E}:\mathbf{f}, \tag{28}$$

where H is the generalized strain-hardening modulus which is positive, zero, or negative for strain-hardening, perfect, and strain-softening plasticity, respectively. Plastic loading occurs when $\Psi = 0$ and $\mathbf{f}:\mathbf{E}:\dot{\varepsilon} > 0$.

Consider the eigensystems of $\mathbf{E}:(\lambda_i, \mathbf{e}_i)$ and $\mathbf{D}^{\mathrm{s}}:(\omega_i, \mathbf{x}_i)$ as presented previously. The elastic tangent modulus tensor, $\mathbf{E}$, is symmetric and positive definite. Also, the eigenvectors for $\mathbf{E}$ span the space of symmetric second order tensors. The eigenvalues and eigenvectors for the symmetric part of the plastic tangent modulus tensor, $\mathbf{D}^{\mathrm{s}}$, will depend on the specific plasticity model being used. Since the eigentensors of $\mathbf{E}$ span the space of symmetric second order tensors, we can express $\mathbf{f}$ and $\mathbf{g}$ as a linear combination of the eigentensors of $\mathbf{E}$. Suppose $\mathbf{f}$ and $\mathbf{g}$ can be expressed as a linear combination of two of the elastic eigentensors (say the first and second to be specific). Then

$$\mathbf{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2, \quad \mathbf{g} = g_1\mathbf{e}_1 + g_2\mathbf{e}_2, \tag{29}$$

and

$$\mathbf{E}:\mathbf{f} = f_1\lambda_1\mathbf{e}_1 + f_2\lambda_2\mathbf{e}_2, \quad \mathbf{E}:\mathbf{g} = g_1\lambda_1\mathbf{e}_1 + g_2\lambda_2\mathbf{e}_2. \tag{30}$$

Postulate an eigentensor for $\mathbf{D}^{\mathrm{s}}$ of the form

$$\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2. \tag{31}$$

Then

$$\mathbf{D}^s:\mathbf{x} = \mathbf{E}:\mathbf{x} - \frac{\mathbf{f}:\mathbf{E}:\mathbf{x}}{2A}\mathbf{E}:\mathbf{g} - \frac{\mathbf{g}:\mathbf{E}:\mathbf{x}}{2A}\mathbf{E}:\mathbf{f} = \omega(\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2), \tag{32}$$

which shows that the postulated form of eqn (31) is valid. After equating the coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$, the result is

$$\begin{bmatrix} (2A\omega - 2A\lambda_1 + 2g_1 f_1 \lambda_1^2) & (g_1 f_2 \lambda_1 \lambda_2 + g_2 f_1 \lambda_1 \lambda_2) \\ (g_2 f_1 \lambda_1 \lambda_2 + g_1 f_2 \lambda_1 \lambda_2) & (2A\omega - 2A\lambda_2 + 2g_2 f_2 \lambda_2^2) \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{33}$$

For a non-trivial solution to exist, the determinant of the coefficient matrix must equal zero. The quadratic characteristic equation yields the two eigenvalues ω_1, ω_2 and the corresponding normalized eigentensors $\mathbf{x}_1$ and $\mathbf{x}_2$ are obtained by determining the associated values of ξ_1 and ξ_2. The remaining four eigenvalues and eigentensors of $\mathbf{D}^s$ coincide with those of $\mathbf{E}$ and span the remaining space of symmetric second order tensors. The procedure can be extended in a similar manner to the case where $\mathbf{f}$ and $\mathbf{g}$ are members of any subspace of the space spanned by the eigentensors of $\mathbf{E}$.

The necessary condition for a general bifurcation is initially satisfied when the fundamental eigenvalue for $\mathbf{D}^s$ is equal to zero. The corresponding bifurcation mode is given by the fundamental eigentensor associated with this zero eigenvalue. The solution to the standard eigenproblem [eqn (32)], with ω set equal to zero, yields the critical hardening modulus, H^{gb}, associated with the first possible bifurcation

$$H^{gb} = \tfrac{1}{2}\left(\sqrt{\mathbf{f}:\mathbf{E}:\mathbf{f}}\sqrt{\mathbf{g}:\mathbf{E}:\mathbf{g}} - \mathbf{f}:\mathbf{E}:\mathbf{g}\right) \tag{34}$$

and the corresponding bifurcation mode is

$$\mathbf{x}_1 = \frac{\mathbf{f}}{\sqrt{\mathbf{f}:\mathbf{E}:\mathbf{f}}} + \frac{\mathbf{g}}{\sqrt{\mathbf{g}:\mathbf{E}:\mathbf{g}}}. \tag{35}$$

Equivalent expressions for the hardening modulus associated with the first possible bifurcation, H^{gb}, have previously been obtained by directly solving eqn (2) (Mroz, 1963; Hueckel and Maier, 1977; Raniecki and Bruhns, 1981). Also, Runesson and Mroz (1989) have solved the generalized eigenproblem

$$\mathbf{D}^s:\mathbf{x}_i = \omega_i \mathbf{E}:\mathbf{x}_i \tag{36}$$

and obtained identical expressions for H^{gb} and $\mathbf{x}_1$. It is important to point out, however, that the standard and generalized eigenproblems will only have an identical fundamental eigentensor when the corresponding fundamental eigenvalue is equal to zero. If the fundamental eigenvalue for $\mathbf{D}^s$ is not equal to zero then the eigensystem for the standard eigenproblem must be obtained to perform the bifurcation analyses.

DRUCKER–PRAGER WITH NON-ASSOCIATED FLOW

Consider a Drucker–Prager plasticity model with a yield function given by

$$\Psi = \sqrt{J_2} + \frac{\mu}{3} I_1 - k, \tag{37}$$

where J_2 is the second invariant of the deviatoric stress and I_1 is the first invariant of the total stress as follows:

$$J_2 = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\sigma}^{\mathrm{d}}, \quad I_1 = \boldsymbol{\sigma}:\mathbf{i}, \tag{38}$$

where $\boldsymbol{\sigma}^{\mathrm{d}}$ is the stress deviator. The normal to the yield surface, $\mathbf{f}$, is given by

$$\mathbf{f} = \frac{\boldsymbol{\sigma}^{\mathrm{d}}}{(2\sqrt{J_2})} + \frac{\mu}{3}\mathbf{i} \tag{39}$$

and the orientation of the plastic strain increment is

$$\mathbf{g} = \frac{\boldsymbol{\sigma}^{\mathrm{d}}}{(2\sqrt{J_2})} + \frac{\beta}{3}\mathbf{i}. \tag{40}$$

The tangent modulus tensor for a plastic step is given by eqn (26).

Unfortunately, an eigenanalysis of the plastic tangent modulus tensor for this material is rather difficult because two of the eigenvalues and two of the eigentensors depend on the hardening modulus. However, using the procedure outlined in a previous section, we can obtain values for the critical hardening moduli for various bifurcation criteria. For any loading, the tensors $\mathbf{f}$ and $\mathbf{g}$ can be written as a linear combination of two of the elastic eigentensors, one from the deviatoric space and one from the spherical space. For example, for uniaxial tension in the x_2 direction, $\mathbf{f}$ and $\mathbf{g}$ can be written as linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_6$ from eqn (24) as follows:

$$\mathbf{f} = f_1\mathbf{e}_1 + f_6\mathbf{e}_6, \quad \mathbf{g} = g_1\mathbf{e}_1 + g_6\mathbf{e}_6. \tag{41}$$

Two of the eigentensors associated with $\mathbf{D}^{\mathrm{s}}$ can also be written as linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_6$ as follows:

$$\mathbf{x}_1 = \gamma_1\mathbf{e}_1 + \gamma_6\mathbf{e}_6, \quad \mathbf{x}_6 = \rho_1\mathbf{e}_1 + \rho_6\mathbf{e}_6. \tag{42}$$

The remaining four eigentensors for $\mathbf{D}^{\mathrm{s}}$ have corresponding eigenvalues of $2G$ and are given by

$$\mathbf{x}_i = \mathbf{e}_i, \quad i = 2, 3, 4, 5. \tag{43}$$

From eqn (33), we obtain the eigenvalues associated with $\mathbf{x}_1$ and $\mathbf{x}_6$. The necessary condition for a general bifurcation is first satisfied when the fundamental eigenvalue, ω_1, obtains a value of zero, which occurs when the generalized hardening modulus obtains a value of

$$H^{\mathrm{gb}} = \tfrac{1}{2}\sqrt{(G+\mu^2K)(G+\beta^2K)} - \tfrac{1}{2}(G+\mu\beta K). \tag{44}$$

Any diffuse or discontinuous bifurcation mode can be written as a linear combination of the eigentensors associated with $\mathbf{D}^{\mathrm{s}}$. For a discontinuous bifurcation to occur in the x_1–x_2 plane, $\dot{\varepsilon}_{13}$, $\dot{\varepsilon}_{23}$ and $\dot{\varepsilon}_{33}$ must all equal zero. The conditions $\dot{\varepsilon}_{13} = 0$ and $\dot{\varepsilon}_{23} = 0$ imply that α_4 and α_5 equal zero in eqn (19) and that the mode is restricted to the following form:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_6\mathbf{x}_6 \tag{45}$$

or

$$\dot{\boldsymbol{\varepsilon}} = (\gamma_1 + \alpha_6\rho_1)\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + (\gamma_6 + \alpha_6\rho_6)\mathbf{e}_6 \tag{46}$$

subject to the following constraint which is needed to satisfy the requirement that $\dot{\varepsilon}_{33}$ equals zero

$$-\frac{1}{\sqrt{6}}(\gamma_1+\alpha_6\rho_1)+\frac{1}{\sqrt{2}}\alpha_2+\frac{1}{\sqrt{3}}(\gamma_6+\alpha_6\rho_6) = 0. \tag{47}$$

The general bifurcation condition [eqn (20)] provides the following additional constraint

$$\omega_1 = -\alpha_2^2\omega_2-\alpha_3^2\omega_3-\alpha_6^2\omega_6. \tag{48}$$

Using the above equations and maximizing ω_1 with respect to the independent variables, α_2, α_3 and α_6, we obtain equations for the fundamental eigenvalue associated with loss of strong ellipticity. Then by iteratively solving the eigenvalue problem and the above equations with monotonically decreasing values for the hardening modulus, we identify the critical hardening modulus for the loss of strong ellipticity.

For this material and loading, Rudnicki and Rice (1975) have obtained the following expression for the hardening modulus associated with a classical discontinuous bifurcation:

$$H^{\text{db}} = E\left[\frac{(\mu-\beta)^2}{18(1-\nu)}-\frac{(\mu+\beta-\sqrt{3})^2}{36}\right]. \tag{49}$$

The critical hardening moduli predicted by the various criteria for uniaxial tension are plotted as a function of μ for various values of β in Fig. 3. Figure 3(a) shows critical values of the dimensionless hardening modulus, (H/E), as a function of the internal friction parameter μ with the dilatancy parameter β fixed at -0.3. Loading is characterized by decreasing values of (H/E). Throughout this paper a value of 0.3 is used for Poisson's ratio. The general bifurcation condition is always reached first for all values of $\mu \neq \beta$. When $\mu = \beta = -0.3$, the general bifurcation and limit point bifurcation criteria coincide as they should for an associated law. Similarly, the strong ellipticity condition is always attained prior to the classical discontinuous bifurcation condition unless $\mu = \beta = -0.3$ where the two discontinuous bifurcation criteria coincide as they should for an associated law. Figures 3(b), (c) show similar results when the dilatancy parameter β is fixed at 0.0 and 0.3, respectively. All of these results indicate that the first general bifurcation point is reached in the hardening regime when the flow is non-associative $\mu \neq \beta$. When the flow is associative, the tangent modulus tensor is symmetric, $\mathbf{D} = \mathbf{D}^s$, and the general bifurcation criterion and the limit point bifurcation criterion both identify the limit point as the first general bifurcation point. Likewise, when the flow is associative, the classical discontinuous bifurcation criterion and the loss of strong ellipticity criterion identify the same point in the softening regime as the first discontinuous bifurcation point. To reach a discontinuous bifurcation point, the material must exhibit either strain-softening or non-associative flow. When the flow is non-associative, the loss of strong ellipticity criterion is satisfied prior to the classical discontinuous bifurcation criterion, as expected.

DRUCKER–PRAGER WITH ASSOCIATED FLOW

To analyse a Drucker–Prager material with associated flow for a general stress path, we can simply repeat the process used in the previous section and let $\mu = \beta$. For an associated flow rule, we find that the fundamental eigenvalue for $\mathbf{D}^s$ obtains a value of zero when H equals zero and that a negative H, strain-softening, leads to a negative fundamental eigenvalue. The fundamental eigentensor corresponding to the zero eigenvalue at the limit point is

$$\mathbf{x}_1 = \mathbf{g} = \frac{\boldsymbol{\sigma}^{\text{d}}}{(2\sqrt{J_2})}+\frac{\mu}{3}\mathbf{i}. \tag{50}$$

Four of the eigentensors of $\mathbf{D}^s$ are orthogonal to $\mathbf{i}$ and $\boldsymbol{\sigma}^{\text{d}}$ and span the remaining deviatoric

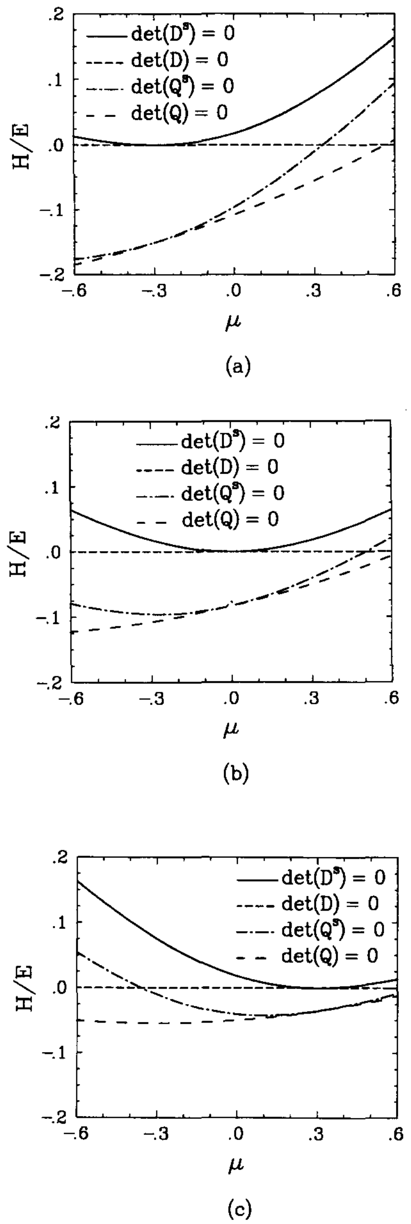


Fig. 3. Critical hardening moduli for a Drucker–Prager material with non-associated flow: (a) $\beta = -0.3$, (b) $\beta = 0.0$, (c) $\beta = 0.3$.

space. These eigentensors have corresponding eigenvalues of $2G$. The remaining eigentensor is given by

$$\mathbf{x}_6 = \mathbf{i} - \frac{\mu \boldsymbol{\sigma}^{\mathrm{d}}}{\sqrt{J_2}} \tag{51}$$

and has a corresponding eigenvalue of

$$\omega_6 = \frac{(3KG + 2\mu^2 KG)}{(G + \mu^2 K)}. \tag{52}$$

Note that when μ obtains a value of zero, the von Mises result of $\mathbf{x}_6 = \mathbf{i}$ and $\omega_6 = 3K$ is obtained. Variations in the hardening modulus, H, lead to changes in ω_1 and ω_6 and their corresponding eigentensors. A negative H leads to a negative fundamental eigenvalue.

The critical hardening moduli predicted by the various criteria for uniaxial tension are plotted as a function of μ in Fig. 4. Note that for an associated flow rule, the general bifurcation and limit point bifurcation criteria both predict the limit point as the first bifurcation point independent of μ. Also, when the flow is associative, the strong ellipticity criterion and the classical discontinuous bifurcation criterion generate identical predictions for the amount of strain-softening needed for localization. For localization to occur, the material must strain-soften or exhibit a significant amount of pressure dependence. Specifically, μ must equal $\sqrt{3}/2$ for a discontinuous bifurcation to occur without strain-softening.

VON MISES

In this section, we analyse a simple von Mises plasticity model with associated flow and a yield function given by

$$\Psi = \sqrt{J_2} - k. \tag{53}$$

Then

$$\mathbf{f} = \mathbf{g} = \frac{\boldsymbol{\sigma}^{\mathrm{d}}}{(2\sqrt{J_2})} \tag{54}$$

and the tangent modulus tensor for a plastic step is given by

$$\mathbf{D} = \mathbf{E} - \frac{2G^2}{(H+G)} \frac{\boldsymbol{\sigma}^{\mathrm{d}} \otimes \boldsymbol{\sigma}^{\mathrm{d}}}{\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\sigma}^{\mathrm{d}}}. \tag{55}$$

The plastic tangent modulus tensor is symmetric, $\mathbf{D} = \mathbf{D}^{\mathrm{s}}$.

The fundamental eigentensor for $\mathbf{D}$ is

$$\mathbf{x}_1 = \frac{\boldsymbol{\sigma}^{\mathrm{d}}}{(\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\sigma}^{\mathrm{d}})^{1/2}} \tag{56}$$

with a corresponding eigenvalue of

$$\omega_1 = 2G\left(\frac{H}{H+G}\right) \tag{57}$$

which varies from $2G$ to 0 as H varies from ∞ to 0 and becomes negative for negative H. The remaining eigentensors for $\mathbf{D}$ are a set of four tensors which span the remaining space

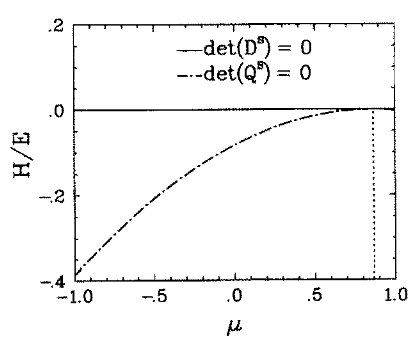


Fig. 4. Critical hardening moduli for a Drucker–Prager material with associated flow.

of symmetric second order deviatoric tensors with corresponding eigenvalues of $2G$ and the second order identity with a corresponding eigenvalue of $3K$. Note that for this model none of the eigentensors and only one of the eigenvalues, the fundamental one, depend on the hardening modulus, H. The fundamental eigentensor does, however, depend on the current stress state.

A general bifurcation may first occur when the fundamental eigenvalue obtains a value of zero. At this point, the hardening modulus, H, equals zero. The character of the bifurcation is given by the fundamental eigentensor. For example, when the material is subjected to uniaxial tension,

$$\mathbf{x}_1 \Rightarrow \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{58}$$

which represents a diffuse bifurcation since $\mathbf{x}_1$ has no zero eigenvalues. If constraints are present, the material may be loaded into a strain-softening regime without activating the first possible general bifurcation. In the strain-softening regime, the fundamental eigenvalue is negative, and alternate bifurcation modes may be activated.

Any bifurcation mode can be written as a linear combination of the eigentensors as given in eqn (19). For a discontinuous bifurcation to occur in the x_1–x_2 plane, $\dot{\varepsilon}_{13}$, $\dot{\varepsilon}_{23}$ and $\dot{\varepsilon}_{33}$ must all equal zero. This implies that α_4 and α_5 equal zero and the mode is restricted to the following form

$$\dot{\boldsymbol{\varepsilon}} \Rightarrow \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{\alpha_2}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\alpha_3}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha_6}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{59}$$

subject to the constraint

$$\frac{-1}{\sqrt{6}} + \frac{\alpha_2}{\sqrt{2}} + \frac{\alpha_6}{\sqrt{3}} = 0, \tag{60}$$

which is an equation for a straight line in the α_2–α_6 plane. Note that α_3 is arbitrary. The general bifurcation condition [eqn (20)] gives the following additional constraint:

$$2G\alpha_2^2 + 2G\alpha_3^2 + 3K\alpha_6^2 = -\omega_1, \tag{61}$$

which is the equation for an ellipse in the α_2–α_6 plane. Note that when ω_1 equals zero, the general bifurcation regime is a single point with $\alpha_2 = \alpha_3 = \alpha_6 = 0$. As H and ω_1 become negative, the size of the general bifurcation regime grows. The first discontinuous bifurcation mode is reached when H obtains the large negative value of $-E/12$. At this point, $\alpha_2 = \sqrt{3}(1+\nu)/(5-\nu)$, $\alpha_6 = \sqrt{2}(1-2\nu)/(5-\nu)$, and $\alpha_3 = 0$ (Fig. 5). Ottosen and Runesson

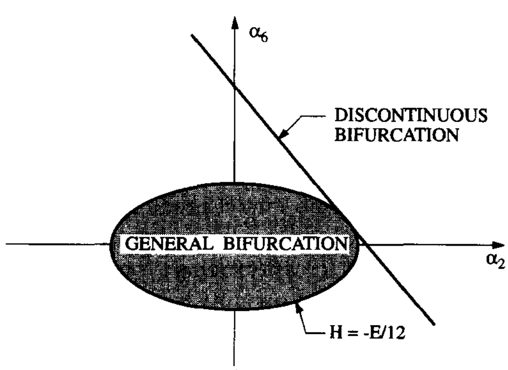


Fig. 5. Bifurcation regimes for a von Mises material subject to uniaxial tension.

(1991) analysed the acoustic tensor and also showed that the first discontinuous bifurcation mode is reached at this point for a von Mises material subjected to uniaxial tension. If the hardening modulus decreases beyond $-E/12$ additional discontinuous bifurcation modes may be activated.

Since the fundamental eigentensor depends on the stress state, the amount of softening needed to reach a discontinuous bifurcation will also depend on the stress state. For example, when the von Mises material is subjected to a pure shear stress in the x_1–x_2 plane, the components of the fundamental eigentensor are

$$\mathbf{x}_1 \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{62}$$

which indicates that the necessary condition for a discontinuous bifurcation is satisfied at the limit point when $H = 0$. The important point here is that by altering the stress path, the amount of strain-softening needed to satisfy the necessary conditions for a discontinuous bifurcation can be significantly changed. This occurs because the fundamental eigentensor for this material model depends on the stress state.

MOHR–COULOMB

Next, consider a simple Mohr–Coulomb model with a yield function given by

$$\Psi = \tfrac{1}{2}(\sigma_{11}-\sigma_{22})+\tfrac{1}{2}(\sigma_{11}+\sigma_{22})\sin\phi - C\cos\phi, \tag{63}$$

where ϕ is the internal friction angle and the coordinate system is chosen such that σ_{11} and σ_{22} are the maximum and minimum principal stresses, respectively. Then for an associated flow law

$$\mathbf{f} = \mathbf{g} \Rightarrow \frac{1}{2}\begin{bmatrix} 1+\sin\phi & 0 & 0 \\ 0 & -1+\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{64}$$

and the tangent modulus tensor for a plastic step is again given by eqn (26). For this model, the bifurcation at the limit point, $H = 0$, is always a discontinuous bifurcation since the fundamental eigentensor for $\mathbf{D}^s$ is equal to $\mathbf{f}$ which has the characteristics required for a discontinuous mode. The nature of the discontinuous bifurcation, as given by the eigentensor, indicates that the components of $\mathbf{n}$ and $\dot{\mathbf{m}}$ are as follows:

$$\mathbf{n} \Rightarrow \begin{Bmatrix} \sqrt{1+\sin\phi} \\ \sqrt{1-\sin\phi} \\ 0 \end{Bmatrix} = c\begin{Bmatrix} 1 \\ \tan\left(\dfrac{\pi}{4}+\dfrac{\phi}{2}\right) \\ 0 \end{Bmatrix}, \quad \dot{\mathbf{m}} \Rightarrow \frac{1}{2}\begin{Bmatrix} \sqrt{1+\sin\phi} \\ -\sqrt{1-\sin\phi} \\ 0 \end{Bmatrix}. \tag{65}$$

It is not surprising that the Mohr–Coulomb model predicts a discontinuous bifurcation at the limit point and that the orientation of the localized zone depends on the internal friction angle because this model was developed to capture this type of failure. A shear localization is obtained when $\phi = 0$ and a discontinuous opening mode is obtained when $\phi = \pi/2$. This analysis indicates that a Mohr–Coulomb model is appropriate for materials that exhibit localization at the limit point with an orientation that is dependent on an internal friction angle.

A Tresca model with associated flow is identical to a Mohr–Coulomb model with the internal friction angle set equal to zero. Thus, the previous results indicate that a Tresca model is appropriate for materials that display localization at the limit point with an orientation of 45° in the plane of maximum and minimum principal stresses. This model does not predict the generation of necking or diffuse bifurcation modes that are sometimes observed experimentally in metals and is, therefore, not appropriate for metals.

EVALUATION OF VON MISES PLASTICITY MODEL FOR METALS

A von Mises plasticity model with associated flow has been used extensively to describe the plastic deformation of metals. This model does an excellent job of capturing the initiation of plastic deformation in metals. In this section, we use the previous analyses to determine if there is any relationship between the bifurcations predicted by this model and experimentally observed necking and localized deformations in metals. Since we have considered only infinitesimal deformations, this comparison between analysis and experiment is only valid for metals which exhibit infinitesimal deformations prior to the initiation of a bifurcation. Several different experimental investigations that may enhance an understanding of the necking and localized deformation in metals are reviewed.

Axisymmetric rod subjected to uniaxial tension

First consider a metallic, axisymmetric rod subjected to uniaxial tension. The first type of bifurcation that is generated in metal rods is necking at some section along the length of the rod. In some rods, for example aluminum rods tested at high temperature (Nadai, 1950), the applied load slowly decreases as the rods continue to neck until the cross-sectional area in the necked region is reduced to a point. In other materials, the initiation of necking is quickly followed by the formation of a crack either at an angle or perpendicular to the applied loading. Here we interpret a crack or the formation of very thin localized deformation zones as evidence of a discontinuous bifurcation, and the initiation of necking as evidence of a diffuse bifurcation.

Needleman (1972), Hutchinson and Miles (1974) and Miles (1975) have all investigated the necking of rods subjected to uniaxial tension using Hill's (1958) general bifurcation criterion. These researchers have all shown that the initiation of necking is coincident with the attainment of maximum load. The necking in the rod is related to activation of the first general bifurcation mode and is characterized by the fundamental eigentensor, $\mathbf{x}_1$ in eqn (58) (Fig. 6). Materials that neck until failure do not strain-soften and therefore do not allow for the activation of any other general or discontinuous bifurcation modes. The fundamental eigentensor associated with the tangent modulus tensor for a von Mises material is dependent

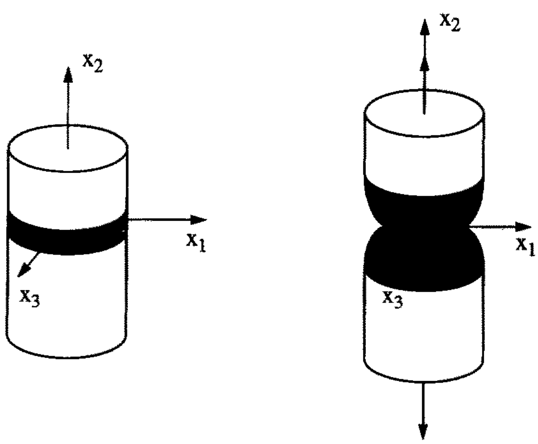


Fig. 6. Activation of the first possible general bifurcation mode in a von Mises rod subject to uniaxial tension.

on the stress state. As the necked region evolves, the stress state in the necked region changes. This leads to a local change in the stress deviator and thus a change in the fundamental eigentensor. Specifically, as necking occurs, a shear stress component is introduced which directly leads to the introduction of a shear strain component in the fundamental eigentensor which explains the formation of a necked region without the generation of any incompatibilities in the neck.

In other materials, a small amount of necking is quickly followed by the formation of a crack either at an angle or perpendicular to the applied loading. The analyses given above indicates that a von Mises material subjected to a pure shear stress can localize without softening but that a von Mises material subjected to a uniaxial tensile stress must exhibit a significant amount of softening to localize. As the necked region evolves in the bar, the stress state deviates from a homogeneous uniaxial tensile stress state and strictly speaking the results from the above analyses do not apply. However, if the amount of necking is not significant, then it seems reasonable to expect the stress state in the necked region to approach that of the original homogeneous uniaxial tensile stress state and to conclude that the material must exhibit strain-softening for the localization to occur. Of course, as shown in the analysis of the Drucker–Prager model, the localization could also be due to pressure dependence of the yield surface, non-associated flow, or a combination of softening, pressure dependence and non-associativity. An accurate study of the evolution of the necked region and subsequent potential localization would require a numerical study similar to that of Needleman (1972) which would allow a characterization of the stress state in the evolving necked region.

Axisymmetric bar with lateral displacements constrained

Next, consider the same axisymmetric bar subjected to uniaxial tension with the artificial constraint that lateral displacements be identical but not necessarily equal to zero along the entire length of the bar as shown in Fig. 7. This constraint will not allow the bar to neck and will thus constrain the first general bifurcation mode. Furthermore, the first discontinuous bifurcation mode given by eqn (59), which is characterized by the formation of a localized zone at an angle of 48.8° from the loading axis will also be constrained. With continued softening of the material, the first unconstrained discontinuous bifurcation that will actually lead to localization is an opening mode discontinuous bifurcation with an orientation, $\mathbf{n}$, parallel to the loading axis. To activate this mode α_6 in eqn (59) must equal $1/\sqrt{2}$ and $H = -E/(6-6\nu)$. This example shows that when the first possible discontinuous bifurcation is constrained, the localization that is ultimately generated is not characterized by the mode associated with the first possible discontinuous bifurcation but rather by the first unconstrained discontinuous bifurcation.

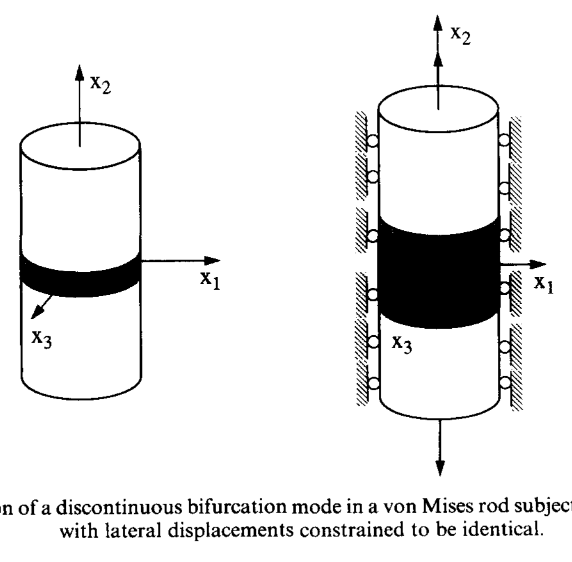


Fig. 7. Activation of a discontinuous bifurcation mode in a von Mises rod subject to uniaxial tension with lateral displacements constrained to be identical.

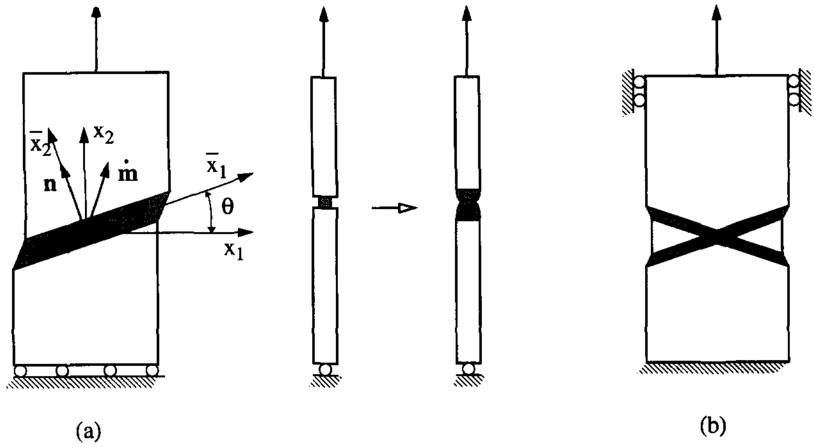


Fig. 8. Activation of the first general bifurcation mode in a von Mises plate subject to uniaxial tension with: (a) no lateral constraint, (b) lateral displacements constrained at the ends.

Thin plate subjected to uniaxial tension

Experiments on thin metal plates subject to uniaxial in-plane tension indicate that necked regions form at angles of between 55 and 65° from the loading axis (Nadai, 1950; Aronofsky, 1951) and not perpendicular to the applied loading as with axisymmetric rods. Again, the formation of the neck is sometimes followed by the formation of a crack plane, and at other times the material just continues to neck until failure.

Necking in a thin plate is again associated with the activation of the first general bifurcation and characterized by the fundamental eigentensor. In the rotated coordinate system shown in Fig. 8, the components of the fundamental eigentensor for $\mathbf{D}^s$ based on eqn (56) are

$$\bar{\mathbf{x}}_1 \Rightarrow \frac{1}{\sqrt{6}}\begin{bmatrix} 2\sin^2(\theta)-\cos^2(\theta) & 3\cos(\theta)\sin(\theta) & 0 \\ 3\cos(\theta)\sin(\theta) & 2\cos^2(\theta)-\sin^2(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{66}$$

For a thin necked region to form in the plane of the plate, the x_1–x_2 plane, and to remain compatible with the surrounding material, the perturbation to the strain rate field in the necked region is subject to the constraint that the 11 components of the fundamental eigentensor in the rotated coordinate system must equal zero. This requirement would be that of a discontinuous bifurcation if the third eigenvalue of $\mathbf{x}_1$ is zero instead of -1. Therefore, there is a potential incompatibility in the x_3 direction. The in-plane constraint yields $2\sin^2(\theta)-\cos^2(\theta) = 0$ or $\theta = 35.3°$. In other words, the predicted necked region is oriented at an angle of 54.7° from the loading axis which corresponds to many experimental observations and is identical to the orientation predicted by Nadai (1950) and Thomas (1961) using similar in-plane compatibility arguments. However, these authors did not address the potential incompatibility in the x_3 direction. The necking is characterized by the fundamental eigentensor which, for a von Mises material, is dependent on the stress state. As the necked region evolves, the stress state in the necked region changes. This leads to a local change in the stress deviator and thus a change in the fundamental eigentensor. The components of the fundamental eigentensor are expected to vary continuously as a function of location within the necked region which explains the formation of the necked region without the generation of any incompatibilities which were introduced and left unexplained in Thomas' (1961) analysis of this thin plate problem.

As shown in Fig. 8(a), the orientation of $\dot{\mathbf{m}}$ which represents the orientation of relative velocities of bodies on opposite sides of the necked region indicates that at least one end of the specimen must be free to move in the x_1 direction for the neck to occur as shown.

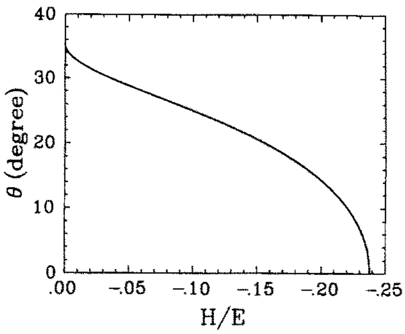


Fig. 9. Effect of strain-softening on the predicted orientation of localization.

Vardoulakis *et al.* (1978) have developed a device that allows for this lateral displacement; however, many specimens are tested in devices that do not allow lateral displacement. This leads to the formation of two necked regions and relative motion of the bodies on opposite sides of the necks as shown in Fig. 8(b).

Aronofsky (1951) shows that the variation in the orientation of the necked region from 25 to 35° could be due to material anisotropy. The analysis presented in the previous sections allows an alternative explanation of this phenomenon if the loading device provides a constraint which allows the material to be loaded into a strain-softening regime. For example, consider the strain perturbation given by

$$\dot{\varepsilon} = \mathbf{x}_1 + \frac{1}{\sqrt{2}}\beta\mathbf{x}_6, \tag{67}$$

which satisfies the general bifurcation condition of eqn (20). Components of the strain perturbation in the rotated coordinate system become

$$\dot{\varepsilon} \Rightarrow \frac{1}{\sqrt{6}}\begin{bmatrix} 2\sin^2(\theta)-\cos^2(\theta)+\beta & 3\cos(\theta)\sin(\theta) & 0 \\ 3\cos(\theta)\sin(\theta) & 2\cos^2(\theta)-\sin^2(\theta)+\beta & 0 \\ 0 & 0 & -1+\beta \end{bmatrix}. \tag{68}$$

The predicted orientation of the necked region is plotted as a function of the amount of softening needed to satisfy the general bifurcation condition in Fig. 9. When the hardening modulus, H, has a value of $-E/10$ an orientation of 25° (65° from the loading axis) is predicted. Even less strain-softening is needed to activate modes with orientations between 25 and 35°. These results suggest that the experimentally observed variations in orientation could be caused by the combination of loading constraints and some strain-softening. Also, when the hardening modulus obtains a value of $-E/(6-6\nu)$ an orientation of 0° and a discontinuous opening mode bifurcation is predicted which corresponds to localization perpendicular to the applied loading. Such a localization is sometimes observed experimentally which would suggest that some metals must exhibit a significant amount of strain softening, pressure dependence or non-associativity at failure.

Thin plates subjected to equal biaxial tension

Metal sheets are often formed by subjecting them to equal biaxial tension with a hemispherical punch. At some point in the forming process intersecting shear bands form at angles through the thickness (Beaver, 1983) as shown in Fig. 10. Forming limit diagrams indicate that thin plates can be plastically deformed significantly farther if they are subjected to equal biaxial tension than if they are subjected to uniaxial tension. The necking that is generated prior to localization in thin plates subject to uniaxial tension is not observed in thin plates subject to equal biaxial tension. Thus, the change in load path to equal biaxial

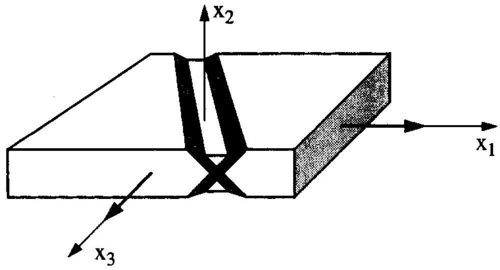


Fig. 10. Localization of a strain-softening, thin plate subject to equal biaxial tension.

tension apparently inhibits the activation of any diffuse bifurcation modes prior to localization.

Storen and Rice (1975) suggest that localization of plates subject to equal biaxial tension provides experimental evidence of the formation of vertices in the yield surface. Recently, Hill (1991) has shown that anisotropic hardening could also lead to the observed localization. Here we present an alternative explanation. For equal biaxial tension in the x_1–x_3 plane (the plane of the plate), the components of the stress tensor and the stress deviator prior to localization are as follows:

$$\boldsymbol{\sigma} \Rightarrow \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\sigma}^{\mathrm{d}} \Rightarrow \frac{\sigma}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{69}$$

For this problem the fundamental eigentensor which is the normalized stress deviator is not compatible in the plane of the plate because the eigenvalues associated with the x_1–x_3 plane are both positive. Thus, no orientation can be found for which a perturbation to the strain rate field by the fundamental eigentensor is compatible in the x_1–x_3 plane with the surrounding material. As the material begins to strain soften, alternate bifurcation modes may be activated. However, by considering all linear combinations of the eigentensors associated with $\mathbf{D}^{\mathrm{s}}$, we quickly find that the first bifurcation mode which satisfies the constraint to remain compatible in the x_1–x_3 plane is a discontinuous bifurcation given by

$$\dot{\boldsymbol{\varepsilon}} \Rightarrow \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\alpha_2}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\alpha_6}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{70}$$

which satisfies the instability condition of eqn (20). This discontinuous bifurcation mode is activated when $H = -E/12$, $\alpha_2 = -\sqrt{3}(1+\nu)/(5-\nu)$ and $\alpha_6 = -\sqrt{2}(1-2\nu)/(5-\nu)$. This bifurcation mode represents shear bands forming at angles of 48.8° from the x_2 axis which is exactly the type of localization that was observed by Beaver (1983). The formation of intersecting shear bands leads to an apparent necking due to the relative motion of material on opposite sides of the shear band (Fig. 10). Instead of the requirement of vertices (Storen and Rice, 1975) or of anisotropic hardening (Hill, 1991), this analysis shows that with a sufficient degree of softening, conventional von Mises plasticity with associated flow can predict the localization in a plate subject to equal biaxial tension. Furthermore, this analysis helps to explain why the necking that is observed in thin plates subject to uniaxial tension is not observed in thin plates subjected to equal biaxial tension.

Thin-walled cylinder subjected to internal pressure

Several investigations [see for example Needleman and Tvergaard (1984)] have involved thin-walled cylinders subjected to internal pressure. These cylinders fail rather catastrophically, with cracks forming along the axis of the cylinder and at an angle through the thickness (discontinuous bifurcation) as shown in Fig. 11.

Prior to localization the components of the stress tensor and the stress deviator are as follows

$$\boldsymbol{\sigma} \Rightarrow \frac{\sigma}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}^{\mathrm{d}} \Rightarrow \frac{\sigma}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{71}$$

We see that the stress deviator is different from the stress deviator found in the previous examples. For this example, the fundamental eigentensor for $\mathbf{D}^{\mathrm{s}}$ which is equal to the normalized stress deviator represents a discontinuous bifurcation mode. Furthermore, the fundamental eigentensor indicates that a shear band oriented at 45° through the thickness and along the axis of the specimen will occur at the limit point. This predicted localization was observed experimentally by Needleman and Tvergaard (1984). Comparing this result with the previous ones, we see that the stress state generated in the wall of a pressurized cylinder has a detrimental effect on the apparent ductility of the material.

These examples show that the type of bifurcation, diffuse or discontinuous, exhibited by metals apparently depends not only on the material but also on the geometry and the prebifurcation stress state. A simple von Mises plasticity model with associated flow predicts both the necking and the localization that is observed experimentally. This analysis indicates that strain-softening can account for many features observed in metals. Most previous analyses have focused primarily on pressure dependence, nonassociativity and vertex development in the yield surface to account for the experimental observations. A close correlation between theoretical predictions and experimental data is needed to provide answers concerning which effect is dominant for any given material.

CONCLUSIONS

Necessary conditions for general and discontinuous bifurcations were reviewed. A necessary condition for the activation of a general bifurcation mode is that the symmetric part of the tangent operator, $\mathbf{D}^{\mathrm{s}}$ is not positive definite. A necessary condition for general bifurcations with compatible bifurcation modes is the loss of strong ellipticity criterion which states that the symmetric part of the acoustic tensor, $\mathbf{Q}^{\mathrm{s}} = \mathbf{n}\cdot\mathbf{D}^{\mathrm{s}}\cdot\mathbf{n}$, is not positive definite. The classical necessary condition for localization is that the acoustic tensor, $\mathbf{Q}$, has a zero eigenvalue. Here we propose that the loss of strong ellipticity criterion which identifies the first possible bifurcation with a kinematically compatible mode should be used as the necessary condition for localization.

The fundamental eigentensor associated with $\mathbf{D}^{\mathrm{s}}$ identifies the character of the first general bifurcation mode. Any diffuse or discontinuous bifurcation mode can be represented

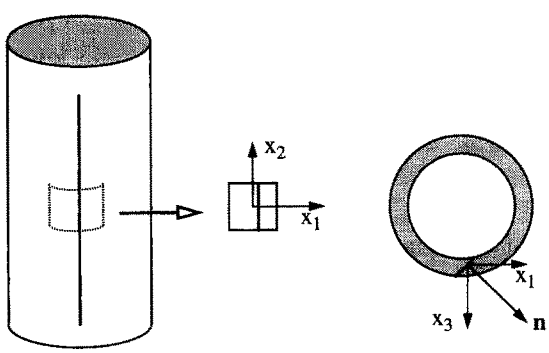


Fig. 11. Localization of a pressurized cylinder.

as a linear combination of the eigentensors for $\mathbf{D}^s$. Necking occurs when the first unconstrained general bifurcation mode is reached. Localization occurs when the first unconstrained discontinuous bifurcation mode is reached. Deformation modes associated with general bifurcations have an inherent potential incompatibility which naturally leads to the evolution of a zone of finite width. Localized deformation zones associated with discontinuous bifurcations have an arbitrary width. A non-local constitutive theory is one approach to generate a zone of finite width and to maintain strong ellipticity when failure is governed by a discontinuous bifurcation.

A review of experimental observations of localized deformation in metals indicates that the type of localized deformation that is generated will depend on the material, loading and geometry. A von Mises plasticity model captures the variation in bifurcation mode with stress state because the fundamental eigentensor associated with the tangent modulus tensor for this model depends on the stress state and a change in the stress state leads to a change in the associated bifurcation mode.

An analysis of Tresca and Mohr–Coulomb models indicates that these models will always predict a discontinuous bifurcation at the limit point. The orientation of the localized deformation zone predicted by these models is rather insensitive to changes in stress state. For example, the Tresca model always predicts a shear band oriented at 45° from the principal stress axes and in the plane of maximum and minimum principal stress. The Tresca and Mohr–Coulomb models are not able to predict the change in bifurcation mode and orientation with changes in stress state and, therefore, do not seem to be appropriate models for metals.

The bifurcation analyses performed in this paper along with experimental observations of localized deformation can be used to evaluate the suitability of constitutive models for various materials. For example, the existence of strain softening rather than non-associativity or pressure dependence in a simple von Mises plasticity model provides the capability for predicting a wide range of features that are observed experimentally in a number of metals. Similar analyses for material models exhibiting different assumptions should be equally revealing.

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